13th International Conference Series on
Laser-light and Interactions with Particles
Optical Particle Characterization follow-up !
August 21-26th, 2022, Warsaw, Poland

Analytical Solutions to Classes of Integrals with Products of Bessel Functions of the First Kind and Their Derivatives

Leonardo André AMBROSIO[1,*], Gérard GOUESBET[2] and Jiajie WANG[3]

[1] Department of Electrical and Computer Engineering. São Carlos School of Engineering, University of São Paulo. 400 Trabalhador são-carlense Ave. 13566-590, São Carlos, SP, Brazil.

[2] CORIA-UMR 6614 - Normandie Université. CNRS-Université et INSA de Rouen. Campus Universitaire du Madrillet. 76800, Saint-Etienne du Rouvray, France.

[3] School of Physics, Xidian University, 710071, Xi'an, China.

Corresponding author: leo@sc.usp.br

Abstract

Recently, attention has been paid to light scattering problems involving photophoresis with arbitrary-shaped beams. In particular, the determination of longitudinal and transverse photophoretic asymmetry factors (PAFs) plays a crucial role in the scattering problem by lossy microparticles, since such factors are proportional to photophoretic optical forces. Here, we show that semi-analytical formulas recently presented for PAFs associated with light scattering by lossy infinite cylinders, relying on integrals with products of Bessel functions of the first kind and their derivatives, can actually be given a complete analytical form, therefore speeding up calculations in light-matter interaction between an incident optical field and a cylindrical scatterer with unbalanced heat absorption. To the best of the authors knowledge, the solutions to such integrals are not available in the literature.

1 Introduction

When a lossy and homogeneous rigid microparticle absorbs light, unbalanced heat absorption takes place that gives rise to thermal forces called photophoretic forces $\mathbf{F}_{ph}$ [1-6]. The determination of such forces is usually accomplished by first finding quantities called (photophoretic) asymmetry factors (PAFs), J_1 [1], or asymmetry vectors, $\mathbf{r}_{as}$ [7]. For instance, under uniform plane wave illumination and assuming propagation along +z, $\mathbf{F}_{ph} = \hat{\mathbf{z}}F_z \propto \hat{\mathbf{z}}J_1$, the proportionality depends on hydrodynamic and thermal properties of the external fluid and the particle [1,7].

In a series of recent works, Mitri explored the interaction between a lossy infinite cylinder and arbitrary-shaped beams and provided expressions for the PAFs using a semi-analytic approach. This approach, based on a partial wave expansion of the internal fields, has been shown to be valid either for isolated dielectric or magnetodielectric cylinders or for cylinders close to planar boundaries and corner spaces [8-10].

A common fact about all the expressions for the PAFs available in these works is the presence of integrals of the following form:

$$I_1^{\pm} = \int_0^1 \xi^2 J_n\left(m_c ka\xi\right) J_{n\pm1}\left(m_c^* ka\xi\right) d\xi \tag{1}$$

and

$$I_2^{\pm} = \int_0^1 \left[\frac{n(n\pm1)}{|m_c|^2 (ka)^2} J_n\left(m_c ka\xi\right) J_{n\pm1}\left(m_c^* ka\xi\right) + \xi^2 J_n'\left(m_c ka\xi\right) J_{n\pm1}'\left(m_c^* ka\xi\right) \right] d\xi \tag{2}$$

In Eqs. (1) and (2), n is an integer ($-\infty \le n \le \infty$), $J_n(x)$ are Bessel functions of the first kind, a is the radius and m_c is the complex relative refractive index of the cylinder with respect to the surroundings, respectively, $k = 2\pi/\lambda$ is the wave number in the external medium, λ being the wavelength. A prime denotes differentiation with respect to the argument, and an asterisk denotes complex conjugation.

Due to the lack of known available analytical solutions to Eqs. (1) and (2), in [8-10] numerical techniques are employed in order to have accurate values of $I_1^{\pm}$ and $I_2^{\pm}$. However, here we show that these integrals can actually be solved after a great deal of algebra.

2 Analytical Expressions for $I_1^{\pm}$ and $I_2^{\pm}$

To solve Eq. (1), we define new variables $\rho = x\xi$, with $x = ka$ being the size parameter of the scatterer. Then, we consider a similar integral, valid for arbitrary real (not necessarily real) v:

$$\frac{1}{x^3}\int_0^x \rho^2 J_v\left(m_c\rho\right) J_{v\pm1}^*\left(m_c\rho\right) d\rho \tag{3}$$

Next, we make use of the recurrence relation $J_{v\mp1}(m_c\rho) = (v/m_c\rho)J_v(m_c\rho) \pm J_v'(m_c\rho)$ to break Eq. (3) into two integrals:

$$\frac{1}{x^3}\int_0^x \rho^2 J_v\left(m_c\rho\right) J_{v\pm1}^*\left(m_c\rho\right) d\rho = \frac{1}{x^3}\frac{v}{m_c^*}\int_0^x \rho\left|J_v\left(m_c\rho\right)\right|^2 d\rho \mp \frac{1}{x^3}\int_0^x \rho^2 J_v\left(m_c\rho\right) J_v'^*\left(m_c\rho\right) d\rho \tag{4}$$

The first integral in the r.h.s. of Eq. (4) can be solved using a similar indefinite integral available in [11], see item 5.54, p. 639, or in [12], see (8), Sec. 5.11 where v is an integer, and one gets:

$$\bar{R}_\nu \equiv \int_0^x \rho \left| J_\nu(m_c\rho) \right|^2 d\rho = \frac{\operatorname{Im}\left[m_c x J_{\nu+1}(m_c x) J_\nu^*(m_c x) \right]}{\operatorname{Im}(m_c^2)} \quad (5)$$

For the second integral in the r.h.s. of Eq. (4), one works out Eqs. (60) and (61) of [5], which are expressed in terms of Ricatti-Bessel functions. After some arrangement and simplification, and performing a substitution of the form $\nu \to \nu - 1/2$, one has

$$\bar{S}_\nu \equiv \int_0^x \rho^2 J_\nu^*(m_c\rho) J_\nu'(m_c\rho) d\rho$$
$$= -\frac{i}{2\operatorname{Im}(m_c^2)} \left\{ x^2 \left[m_c \left| J_\nu(m_c x) \right|^2 + m_c^* \left| J_{\nu+1}(m_c x) \right|^2 \right] \right. \quad (6)$$
$$\left. -2\left[m_c + \nu \frac{\operatorname{Re}(m_c^2)}{m_c} \right] \bar{R}_\nu + 2\nu m_c^* \bar{R}_{\nu+1} \right\}$$

Imposing $\nu = n$, substituting Eqs. (5) and (6) in (4) and after some simplification, one finally arrives at

$$I_1^\pm = \frac{1}{x^3}\left[\frac{n}{m_c^*} \bar{R}_n \mp (\bar{S}_n)^* \right] \quad (7)$$

Equation (7) shows that a full analytical solution can be given to Eq. (1). Similarly, one can also solve for Eq. (2). Although straightforward, we shall here omit the details, which involves the use of recurrence relations to break into parts and rewrite the term with products between derivatives of Bessel functions in the second line of Eq. (2). One then performs some rearrangements and identifies each new integral either with either $\bar{R}_\nu$, $\bar{S}_\nu$ or $I_1^\pm$. After some algebra and simplification, one can then show that:

$$I_2^\pm = \frac{1}{x^3}\left(\frac{n \pm 1}{m_c^*} \bar{R}_{n\pm1} \pm \bar{S}_n \right) \quad (8)$$

The validity of Eqs. (7) and (8) have been assessed by running computer simulations. They have been compared with the integrals with Eqs. (1) and (2) by implementing them using the commercial software *Wolfram Mathematica 12.1 Student Edition*. The code was then run on a personal computer [Intel(R) Core(TM) i7-3630QM CPU @ 2.40GHz, 16.0 GB]. Elapsed times for each case have also been computed and saved for comparison.

The solutions provided by Eqs. (7) and (8) can be used to evaluate the PAFs for infinite homogenous cylinders. As we intend to show during the conference, divergences between Eqs. (1), (2) and Eqs. (7), (8) are attributed to numerical errors and can be made smaller by increasing the numerical precision. Computational burden is reduced by approximately two orders of magnitude.

3 Conclusions

We have here derived exact and analytical solutions to integrals with products of Bessel functions of the first kind and their derivatives which, to the best of the authors knowledge, are not available anywhere in the literature. Such solutions are of special importance in light scattering problems by lossy and homogenous infinite cylinders, mainly in the determination of photophoretic asymmetry factors. However, it is expected that the present work serves as a first step towards new analytical solutions to classes of integrals involving products of Bessel functions of different kinds and their derivatives, as happens to be observed when the cylinder is not electromagnetically homogeneous. A similar problem might occur for multi-layered/coated spheres, and the results here presented can certainly be of help in putting aside semi-analytical methods in the analysis of photophoretic forces and asymmetry factors for arbitrary shaped beams in generalized Lorenz-Mie theories.

4 Acknowledgement

L. A. Ambrosio thanks the National Council for Scientific and Technological Development (CNPq) (426990/2018-8, 309201/2021-7) for supporting this work.

5 References

[1] Yalamov Y.I. Kutukov V., Shchukin E., Theory of the photophoretic motion of a large-size volatile aerosol particle, J. Colloid Interface Science 57(3):564-571 (1976)

[2] Reed L.D., Low Knudsen number photophoresis, J. Aero. Sci. 8(2): 123-131 (1977)

[3] Pluchino A.B., Photophoretic force on particles for low Knudsen number, Appl. Opt. 22(1):103-106 (1983)

[4] Greene W.M., Spjut R.E., Bar-Ziv E., Sarofim A.F., Longwell J.P., Photophoresis of irradiated spheres: absorption centers, J. Opt. Soc. Am. B 2(6): 998-1004 (1985)

[5] Mackowski D., Photophoresis of aerosol particles in the free molecular and slip-flow regimes, Intl. J. Heat Mass Trans. 32(5): 843-854 (1989)

[6] Horvarth H., Photophoresis – a Forgotten Force ??, KONA Powder Part. J. 31:181-199 (2014)

[7] Zulehner W., Rohatschek H., Representation and calculations of photophoretic forces and torques, J. Aero. Sci. 26(2): 201-210 (1995)

[8] Mitri F.G., Photophoretic asymmetry factors for an absorptive dielectric cylinder near a reflecting planar boundary, J. Opt. Soc. Am. A 38(12):1901-1910 (2021)

[9] Mitri F.G., Longitudinal and transverse PAFs for an absorptive magneto-dielectric circular cylinder in light-sheets of arbitrary wavefronts and polarization, Appl. Opt. 60(26): 7937-7944 (2021)

[10] Mitri F.G., Effect of a perfectly conducting corner space on the PAFs for an absorptive dielectric circular cylinder, J. Opt. Soc. Am. B 38(12):2910-2919 (2021)

[11] I.S. Gradshteyn, and I.M. Ruzhik, *Table of Integrals, Series and Products* (Academic Press, Burlington, MA, USA, 2007).

[12] Watson G.N., *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, UK, 1944.