BEAM SHAPE COEFFICIENTS OF ELECTROMAGNETIC ZERO-ORDER ON-AXIS CONTINUOUS FROZEN WAVES IN THE GENERALIZED LORENZ-MIE THEORY

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Abstract

Extending a recent theoretical work from the author, it is shown here that it is indeed possible to analytically describe the beam shape coefficients (BSCs) of an interesting and promising type of optical field known as zero-order *continuous frozen wave* (CFW), at least under an on-axis configuration. This represents a first step towards a full analytical incorporation of CFWs in the framework of the generalized Lorenz-Mie theory (GLMT). As will be seen, such a construction demands several intermediate steps which might also be useful for describing azimuthally-symmetric beams, whenever its spectrum – in terms of the longitudinal wave number – is known *a priori*.

1 Brief Introduction and Theoretical Aspects

An optical CFW is a continuous sum of propagating (non-evanescent) Bessel beams. Omitting a time-harmonic convention exp($+i\omega t$), where ω is the angular frequency, and assuming $+z$ free space propagation, its scalar version (which serves as a building block for the construction of vector beams) is written as [1-3]:

$$\psi(\rho,z)=\psi_0\int_{-k}^{k}S(k_z)J_0(k_\rho\rho)e^{-ik_z z}dk_z \tag{1}$$

In Eq. (1), $k = 2\pi/\lambda$ is the wave number, with λ being the wavelength, k_z and k_r are the longitudinal and transverse wave number, respectively and $J_0(.)$ is the zero-order Bessel function. Cylindrical coordinates (ρ,ϕ,z) are assumed attached to both a Cartesian (x,y,z) and a spherical (r,θ,ϕ) coordinate system. In the GLMT, the origin $\mathcal{O}$ of the (x,y,z) system coincides with the centre of a spherical scatterer [4].

A zero-order CFW is therefore a continuous superposition of Bessel beams. It is, however, a superposition of a very special kind, since its spectrum $S(k_z)$ is made attached to its pre-defined longitudinal intensity pattern $|F(z)|^2$ which, in turn, is defined within $-L/2 \le z \le L/2$. Indeed, by expanding $S(k_z)$ into a truncated Fourier series, $S(k_z)=\sum_{l=-l_{\max}}^{l_{\max}}A_l\exp\left(i\pi l k_z/k\right)$, the expansion coefficients A_l are given by [1,2]:

$$A_l=\frac{1}{2k}F\left(\frac{\pi l}{k}\right) \tag{2}$$

that is, the expansion coefficients are sampled values of the reference function $F(z)$, which is a very convenient property.

In a recent work [3], the authors have shown that a full analytical description of the BSCs of vector versions of Eq. (1), with different polarizations and for arbitrary off-axis configuration, would be possible if: (i) certain special functions, viz.,

$$\begin{aligned}
b^{1,2}(k_z)&=m\frac{k_z}{k}\pi_n^m\left(\frac{k_z}{k}\right)\pm\tau_n^m\left(\frac{k_z}{k}\right)\\
b^{3,4}(k_z)&=m\pi_n^m\left(\frac{k_z}{k}\right)\pm\frac{k_z}{k}\tau_n^m\left(\frac{k_z}{k}\right)\\
b^{5}(k_z)&=\frac{k_z}{k}\tau_n^m\left(\frac{k_z}{k}\right)
\end{aligned} \tag{3}$$

can be Fourier expanded, and (ii) integrals of the form

$$\begin{aligned}
I_{k_z}^{j}&=\int_{-k}^{k}dk_z\frac{S(k_z)}{k_z/k}e^{ik_z z_0}b^j(k_z)J_{m\mp1}\left(\rho_0\sqrt{k^2-k_z^2}\right)e^{-i(m\mp1)\phi_0}\\
I_{k_z}^{5}&=\int_{-k}^{k}dk_z\frac{S(k_z)}{k_z/k}e^{ik_z z_0}b^5(k_z)J_{m}\left(\rho_0\sqrt{k^2-k_z^2}\right)e^{-im\phi_0}
\end{aligned} \tag{4}$$

can be analytically evaluated. Both problems have been left open in [3].

In Eq. (3), the upper and lower signs in the l.h.s. refer to the first or the second of the upper indices in the r.h.s, respectively. Besides, $n \ge 1$ and $-n \le m \le n$, with n and m integers, and $\tau_n^m(x)=-(1-x^2)^{1/2}dP_n^m(x)/dx$ and $\pi_n^m(x)=P_n^m(x)/(1-x^2)^{1/2}$ are generalized Legendre functions [4], with $P_n^m(x)$ being the associated Legendre polynomials according to Hobson's definition [5]. In Eq. (4), j = 1 to 4, and one considers that the original scalar beam in Eq. (1), after being taken as one of the transverse electric field components depending on the chosen polarization, is subsequently and arbitrarily displaced by (ρ_0,ϕ_0,z_0) from the origin $\mathcal{O}$.

The TM (Transverse Magnetic) and TE (Transverse Electric) BSCs, $g_{n,TM}^m$ and $g_{n,TE}^m$, are then written in terms of Eq. (4). As an example, for x-polarized CFWs, it can be shown that [3]:

$$g_{n,TM}^m=\frac{i^{m+1}}{2}(-1)^{(m-|m|)/2}\frac{(n-m)!}{(n+|m|)!}\left(I_{k_z}^1+I_{k_z}^2\right), \tag{5}$$

$$g_{n,TE}^m=-i\frac{i^{m+1}}{2}(-1)^{(m-|m|)/2}\frac{(n-m)!}{(n+|m|)!}\left(I_{k_z}^3-I_{k_z}^4\right), \tag{6}$$

with similar expressions for y-, circular, azimuth and radial polarizations.

If, at one hand, it might be challenging to find explicit analytical solutions to Eq. (4) for off-axis CFWs, the situation becomes tractable for $\rho_0 = 0$, that is, for on-axis beams. Under this configuration, the Bessel functions appearing in Eq. (4) are either 1 (when $m = \pm 1$, for $j = 1$ to 4, or when $m = 0$, for $I_{k_z}^5$) or 0. Under such conditions, the exponentials containing ϕ_0 disappear, and the integrals can be readily evaluated, thus allowing one to write, with the aid of Fourier expansions for $S(k_z)$ and $b^j(k_z)$, explicit analytical expressions to Eqs. (5) and (6):

$$g_{n,TM}^{m} = -\frac{i^m}{2}(-1)^{(m-|m|)/2}\frac{(n-m)!}{(n+|m|)!}\sum_{l=-l_{\max}}^{l_{\max}}\sum_{p=-\infty}^{\infty}2kA_l \times\left(B_p^1\delta_{m,1}+B_p^2\delta_{m,-1}\right)\mathrm{Si}\left[\pi(l+p)+kz_0\right], \tag{7}$$

$$g_{n,TE}^{m} = -\frac{i^{m+1}}{2}(-1)^{(m-|m|)/2}\frac{(n-m)!}{(n+|m|)!}\sum_{l=-l_{\max}}^{l_{\max}}\sum_{p=-\infty}^{\infty}2kA_l \times\left(B_p^3\delta_{m,1}+B_p^4\delta_{m,-1}\right)\mathrm{Si}\left[\pi(l+p)+kz_0\right]. \tag{8}$$

In Eqs. (7) and (8), Si[.] is the sine integral function and $\delta_{i,j}$ is the Kronecker delta. In addition, B_p^j (j = 1 to 4) are the Fourier coefficients of the functions defined according to Eq. (3), and they read as:

$$B_p^1 = \frac{n(n+1)}{2}\sum_{q=0}^{Q(n,0)} a_q^{n,0} \times\begin{cases} -B(\mu_1,\nu_1)\,{}_1F_2\left(\nu_1;\tfrac{1}{2},\mu_i+\nu_1;-\tfrac{\pi^2p^2}{4}\right), & n \text{ even} \\ i\pi pB(\mu_1,\nu_1+\tfrac{1}{2})\,{}_1F_2\left(\nu_1+\tfrac{1}{2};\tfrac{3}{2},\mu_1+\nu_1+\tfrac{1}{2};-\tfrac{\pi^2p^2}{4}\right), & n \text{ odd}\end{cases} \tag{9}$$

$$B_p^2 = \frac{1}{n(n+1)}B_p^1, \qquad B_p^3 = B_p^{31}+B_p^{32}, \qquad B_p^4 = \frac{1}{n(n+1)}B_p^3, \tag{10}$$

$$B_p^{3j} = -\frac{1}{2}\sum_{q=0}^{Q(n,j)} a_q^{n,j} \times\begin{cases} -B(\mu_{3j},\nu_{3j})\,{}_1F_2\left(\nu_{3j};\tfrac{1}{2},\mu_{3j}+\nu_{3j};-\tfrac{\pi^2p^2}{4}\right), & n \text{ odd} \\ i\pi pB(\mu_{3j},\nu_{3j}+\tfrac{1}{2})\,{}_1F_2\left(\nu_{3j}+\tfrac{1}{2};\tfrac{3}{2},\mu_{3j}+\nu_{3j}+\tfrac{1}{2};-\tfrac{\pi^2p^2}{4}\right), & n \text{ even}\end{cases} \tag{11}$$

$$B_p^5 = -\frac{1}{2}\sum_{q=0}^{Q(n,1)} a_q^{n,1} \times\begin{cases} -B(\mu_5,\nu_5)\,{}_1F_2\left(\nu_5;\tfrac{1}{2},\mu_5+\nu_5;-\tfrac{\pi^2p^2}{4}\right), & n \text{ even} \\ i\pi pB(\mu_5,\nu_5+\tfrac{1}{2})\,{}_1F_2\left(\nu_5+\tfrac{1}{2};\tfrac{3}{2},\mu_5+\nu_5+\tfrac{1}{2};-\tfrac{\pi^2p^2}{4}\right), & n \text{ odd}\end{cases} \tag{12}$$

In Eqs. (9)-(12), $Q(n,j)$ is the integer part of $(n-j)/2$, $B(\mu,\nu)$ is the Beta function and ${}_1F_2(\alpha;\beta_1,\beta_2;x)$ is the generalized hypergeometric function. In addition, $\mu_1 = \mu_{32} = q+1$, $\mu_{31} = q+2$, $\mu_5 = (2q+3)/2$, $\nu_1 = \nu_5 = (n-2q+1)/2$, $\nu_{31} = (n-q)/2$ and $\nu_{32} = (n-q+2)/2$. Finally, $a_q^{n,j}$ are coefficients which depend on the values of q, n and j, see Eq. (3) of [6].

2 Computational Examples

As an example of computation of CFWs in the GLMT, we consider a x-polarized field with a reference function $F(z) = \exp[-8(2z/L)^2]\cos(8\pi z/L)\exp(-iQz)$, where L = 44 µm. Here, $Q = 0.75k$ determines the degree of paraxiality of the beam, its transverse field concentration and the central position of $S(k_z)$. For the simulations, λ = 1064 nm, free space propagation is assumed and we have truncated the sum over p in Eqs. (7) and (8) at $p = p_{\max} = 100$. Field reconstructions in the GLMT are in accordance with Wiscombe's criterion [7], with expressions for the electromagnetic field components in terms of partial wave expansions available elsewhere, see Eqs. (3.39)-(3.50) of [4].

Figure 1(a) shows $S(k_z)$ for the chosen reference function $F(z)$, while Figure 1(b) illustrates $|\psi(0,z)|^2$ and $|F(z)^2|$. In Eqs. (7) and (8), $l_{\max}$ = ceil[L/λ], with ceil[.] denoting the ceiling function, see [1] for details. It is seen that, for the chosen parameters, Eq. (1) adequately reproduce the intended $F(z)$. Density plots of the expected electric field intensities $|E_x(\rho,z)|^2$ and $|E_z(\rho,z)|^2$ at the xz plane are shown in Figure 2.

The electric field components reconstructed from the GLMT expressions, along ρ = 0, can be appreciated in Figure 3(a). The curve for $|\psi(0,z)|^2$ is again shown for reference purposes. As observed from Figure 3(b) from the logarithmic error $\ln\left[\left|\left(|\psi(0,z)|^2-|E_x(0,z)|^2\right)/|\psi(0,z)|^2\right|\right]$, an excellent field reconstruction is achieved, at least along ρ = 0, which is the region of most interest for practical applications. It can be shown, by plotting $|\psi(\rho,z=z')|^2$ for distinct z', that an excellent agreement is also observed in the transverse direction.

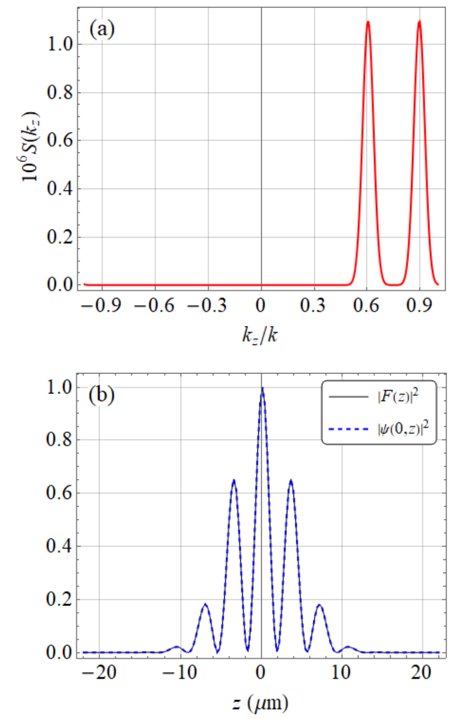

Figure 1 *(a) $S(k_z)$ as a function of k_z for the given reference function $F(z)$. (b) $|\psi(0,z)|^2$ and $|F(z)^2|$, the former being calculated from Eq. (1)*

3 Conclusions

An analytical method for the evaluation of the beam shape coefficients of zero-order continuous frozen waves has been presented under an on-axis configuration. This is a first step towards a full incorporation of such beams in the generalized Lorenz-Mie theory envisioning light scattering applications. Interestingly enough, the method here presented is also suitable for describing on-axis azimuthally-symmetric arbitrary-shaped beams. In this case, however, there is no direct relation between the Fourier coefficients of their spectra and a pre-chosen reference function as expressed here in Eq. (2). This means that their spectra must be known a priori for each particular shaped beam, which might not always be in terms of as shown by the coefficients in Eq. (2). Continuous frozen waves, being a class of micrometer-structured non-diffracting beams, can be of interest in applications ranging from optical tweezers and bistouries, 2D and 3D imaging and printing, holography and so on, and an extension of the present work for arbitrary-order beams under off-axis configuration is certainly deserved. This is current in progress

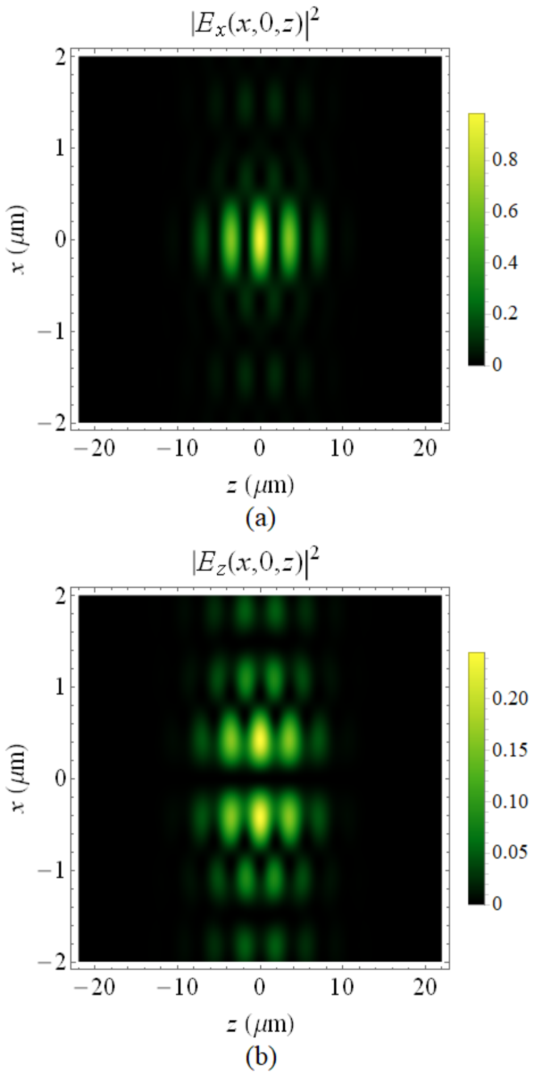

Figure 2 *Density plot of (a) $|E_x(\rho,z)|^2$ and (b) $|E_z(\rho,z)|^2$ at the xz plane, corresponding to the reference function $F(z)$ of Figure 1(b)*

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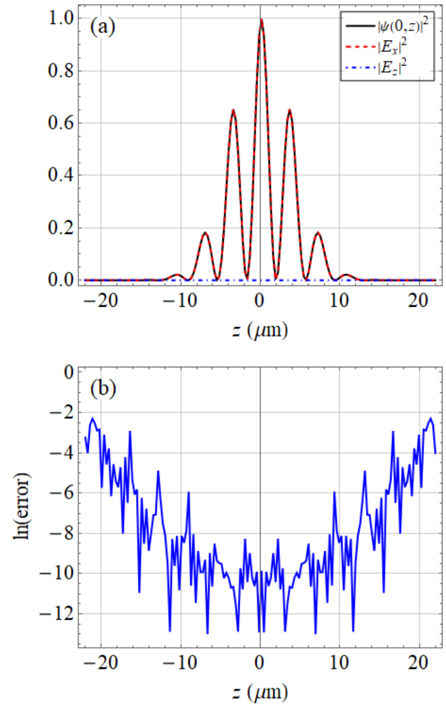

Figure 3 *(a) Electric field components along the z axis, reconstructed using the GLMT [4]. (b) Corresponding logarithmic error* $\ln\left[\left|\left(|\psi(0,z)|^2-|E_x(0,z)|^2\right)/|\psi(0,z)|^2\right|\right]$

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